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Combinatorial complexity of translating a box in polyhedral 3-space [☆]

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Abstract

We study the space of free translations of a box amidst polyhedral obstacles with n vertices. We show that the combinatorial complexity of this space is $O(n^2\alpha(n))$, where $\alpha(n)$ is the inverse Ackermann function. Our bound is within an $\alpha(n)$ factor off the lower bound, and it constitutes an improvement of almost an order of magnitude over the best previously known (and naive) bound for this problem, $O(n^3)$. For the case of a convex polygon of fixed (constant) size translating in the same setting (namely, a two-dimensional polygon translating in three-dimensional space), we show a tight bound $\Theta(n^2\alpha(n))$ on the complexity of the free space. © 1998 Elsevier Science B.V.

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1. Introduction

For over a decade, robot motion planning has attracted much research in various fields and has become a central topic in robotics. The basic motion-planning problem, sometimes referred to as *the piano movers' problem*, is defined as follows.

Let B be a robot system having k degrees of freedom and free to move within a two- or three-dimensional domain V which is bounded by static obstacles whose geometry is known to the system. The motion-planning problem for B is, given the initial and desired final placements of

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the system B, to determine whether there exists a continuous motion from the initial placement to the final one, during which B avoids collision with the known obstacles, and if so, to plan such a motion.

In this pure formulation of the problem, we are only interested in the geometric aspects of the motion. We ignore many issues, such as acceleration, speed, uncertainty or incompleteness in the geometric data, control strategies for executing the motion, etc. A comprehensive overview of problems and techniques in robot motion planning can be found in [14]. Several surveys on the topic have also been published, e.g., [17,18].

One approach to solving motion-planning problems, so-called *exact motion planning*, is non-heuristic. It aims to find a solution whenever one exists and otherwise report that no solution exists. Much of the study of exact motion planning is carried out in the *configuration space* of the problem. The configuration space of a motion-planning problem with k degrees of freedom is k-dimensional and every point in it represents a possible placement of the robot in the physical space.

A fundamental problem here is to understand the *combinatorial complexity* (see definition in the next paragraph) of the underlying configuration space. Such complexity analysis is often a prelude to efficient algorithms, since many motion planning algorithms compute the configuration space or portions thereof.

Throughout this paper, we assume that B (the robot) is a fixed convex rigid body to be moved amidst an *obstacle set* Ω, where B and Ω have piecewise linear boundary. Depending on the context, the underlying physical space is $\mathbb{R}^2$ or $\mathbb{R}^3$. The parameter n denotes the combinatorial complexity of Ω, that is, n is the number of corners, edges and (where applicable) faces of Ω. We will often use the term *feature* to refer to a corner, an edge, or a face of a polyhedral set.

Our new results concern translation in $\mathbb{R}^3$. We assume that the obstacle set $\Omega \subseteq \mathbb{R}^3$ is a "regular" open set, i.e., the set Ω equals the interior of its closure. For any translation $Z \in \mathbb{R}^3$, let $B[Z] \subseteq \mathbb{R}^3$ denote the *position* of B when translated by Z. More precisely, we define $B[Z]$ to be the set $Z + B = \{Z + p\colon\ p \in B\}$. We say that $Z \in \mathbb{R}^3$ is *free* if $B[Z] \cap \Omega = \emptyset$. Let $\mathrm{FP} \subseteq \mathbb{R}^3$ be the space of *free translations* of B amidst Ω. Since in our paper Ω, B are polyhedral, FP is also polyhedral.

The goal is to analyze the combinatorial complexity of FP. We will usually say "complexity" instead of "combinatorial complexity". Some of our bounds will refer to the slow-growing function $\alpha(n)$, the inverse of Ackermann's function.

Using standard arguments relating to certain arrangements of curves or surfaces induced by these motion planning problems, it can be shown that the complexity of the free space, in our setting, for a motion planning problem with two (respectively three) degrees of freedom is $O(n^2)$ (respectively $O(n^3)$). We assume here and throughout the paper that the complexity of the robot is a fixed constant. There are motion planning problems, involving non-convex moving objects, for which these bounds are tight. A major effort in the study of motion planning in computational geometry is devoted to identifying situations where significantly better bounds can be proved. Indeed, when the moving object is convex it is often the case that improved bounds can be obtained.

The planar case is quite well understood. Kedem et al. [11] obtain a linear bound on the complexity of the free space of a convex polygonal robot translating in a two dimensional polygonal space. In fact, their result is more general and concerns the complexity of the boundary of the union of special planar figures. This latter result, which was motivated by a motion planning problem, has found many other applications in computational geometry. Still in the plane, when general rigid motion (translation and rotation) is allowed (so now, the robot has three degrees of freedom), the complexity of the free space for a convex polygonal robot has been shown to be only near-quadratic [15], and near-quadratic time algorithms were devised to solve the motion planning problem [4,12,13]. In summary, for a convex polygonal robot moving among polygonal obstacles in the plane, the bounds on the complexity of the free space were shown to be roughly an order of magnitude lower than the corresponding naive bounds.

Until recently, results for 3-space were quite partial. A natural starting place is the case of pure translations, which is the object of this work. Besides its intrinsic interest, translational motion also arises as a key subproblem in planning general motion. It was conjectured for a long time (see [16]) that the complexity of FP for a convex polyhedron of constant size translating in polyhedral 3-space among obstacles with a total of n features is $O(n^2\alpha(n))$. This conjectured bound is the best possible; see [2] for a construction where the size of FP is $\Omega(n^2\alpha(n))$. Prior to the original publication of our result [8] the only non-trivial result in support of the conjecture has been the case where B is a ladder (line segment). This bound is described in [16], and independently observed by Ke and O'Rourke [10]. Their bound is slightly better than the general conjecture: the factor of $\alpha(n)$ is not needed. We focus on the case of translating a box (i.e., a convex polyhedron with 8 vertices and 6 rectangular faces). The main result in this paper lends further support to the following conjecture.

Theorem 4.5. *For a box B, the complexity of* FP *is* $O(n^2\alpha(n))$.

We also show the following theorem.

Theorem 5.3. *For a convex polygon P of constant size, the complexity of* FP *is* $\Theta(n^2\alpha(n))$.

After our result for a box had originally appeared [8], Aronov and Sharir [2] obtained a near-quadratic bound on the complexity of the free space for an arbitrary convex polyhedron translating among polyhedral obstacles in 3-space. Their result, applied to our special setting (a *box* translating among polyhedral obstacles with a total of n vertices), gives a bound of $O(n^2 \log^2 n)$ on the complexity of the free space, in the worst case. This result was later improved [3] to $O(n^2 \log n)$. Still, in the worst case, our bound is sharper for the special case of a box. We note that our technique is considerably simpler than the techniques of [2,3].

A different approach to motion planning seeks to represent only a single component of the free space. Along these lines, a general near-optimal result on the configuration space of any (reasonable) robot system with 2 degrees of freedom has been shown [5], and for any system with 3 degrees of freedom in [7]. In higher dimensions, Aronov and Sharir [1] show that the complexity of a single cell in an arrangement of n $(d-1)$-simplices in d-space is $O(n^{d-1}\log n)$; rephrased in motion planning terms: in a system with d degrees of freedom, such that all constraints are piece-wise linear and can be described as a union of n simplices, a single connected component has complexity $O(n^{d-1}\log n)$. Of course, this result speaks to our setting in Theorem 4.5 as well, implying an upper bound of $O(n^2 \log n)$. Our result (for B a box in 3-space), however, applies to the *entire* free space FP.

The rest of this paper is organized as follows. In Section 2, we establish some terminology and give a warm-up result. In Section 3, we study a special situation of translating a triangle in space. This will be a critical case when we study boxes. In Section 4, we prove our main result, the bound for translating a box. In Section 5, we present a tight bound for the case of a convex polygon. Some concluding remarks and open problems are given in Section 6.

2. Preliminaries

In this section we establish some terminology and obtain a bound $\Theta(n^2)$ for the case of a rectangle translating among lines in space.

We assume, without loss of generality, that the box B translates so that its edges are parallel to the coordinate axes. We also assume that the box and the obstacles are in *general position*. In particular, we assume that no obstacle edge is parallel to any coordinate axis, that no obstacle face is orthogonal to any coordinate axis, that no three obstacle corners are collinear, etc. It can be shown that such degenerate situations may only decrease the complexity that we aim to bound from above. For detailed discussions on general position assumptions in related motion planning problems, see, e.g., [6,15].

There are two types of points in the configuration space: *free* points, that represent placements of B where it does not intersect any obstacle, and *forbidden* points, that represent placements of B where it penetrates an obstacle. Among the free points we distinguish a subclass of *semi-free* points that represent placements where B touches the boundary of an obstacle but does not penetrate any of the obstacles. The collection of semi-free points in the configuration space forms a collection of polygons that, roughly speaking, separate the free portions of the configuration space from the forbidden portions. We measure the complexity of the free space by the number of features (vertices, edges and faces) showing up on its boundary.

To describe the obstacles in the configuration space, we compute the Minkowski (vector) sum of each obstacle polyhedron and the box, where the box has its center in the origin. [3] It is easily verified that a configuration space obstacle, which is an original obstacle "expanded" by the box B, is also a polyhedron. We refer to configuration space obstacles as *expanded obstacles*.

The features showing up on the boundary of FP represent semi-free positions of the box B. We remind the reader that by a *feature* of a polyhedral set, we mean a face, an edge or a vertex of the set. A face on the boundary of FP is induced by a (semi-free) contact of a feature of B and a feature of an obstacle. An edge on the boundary of FP is induced by a pair of contacts, and a vertex on the boundary of FP is induced by a triple of contacts. Four or more simultaneous contacts are ruled out by the general position assumption. A *(potential) contact* is a pair $O = (x, X)$, where x is a feature of B and X a feature of Ω. We say that a translation Z (not necessarily free) *satisfies* a contact O if $x[Z] \cap X$ is a single point ξ and Z is *locally free* at ξ. (Z is locally free at ξ means that in a small enough neighborhood of ξ, $B[Z]$ intersects the closure of Ω only at ξ.) We need to focus on three types of contacts (below we explain how other types of contacts are handled):

- *edge contact* $O = (e, E)$—an edge e of the box B is in contact with an edge E of an obstacle;
- *face contact* $O = (f, C)$—a face f of B touches an obstacle corner C;
- *corner contact* $O = (c, F)$—a corner c of B touches an obstacle face F.

Let G be a set of contacts. We say that G is *realizable* if there exists a (not necessarily free) translation Z that simultaneously satisfies each $O_i \in G$. If $|G| = 2$ (respectively $|G| = 3$), we call G a *double* (respectively *triple*) *contact*. We usually write $G = (O_1, O_2)$ or $G = (O_1, O_2, O_3)$ for a double or triple contact. By the general position assumption, each O_i in a triple contact G is independent in the

[3] The Minkowski sum of two spatial sets A and B, $A \oplus B$, is the set $\{p + q \mid p \in A,\ q \in B\}$. We are actually interested in the Minkowski sum of each obstacle and $-B$. However, since we place the center point of B at the origin, we have $B = -B$.

sense that if Z realizes G then there are placements in every open neighborhood of Z that satisfy two but not the third contact. If $Z \in \mathrm{FP}$ and Z realizes a triple contact G then Z induces a vertex on the boundary of FP, and we call G a *semi-free* triple contact.

It is important to note that semi-free triple contacts are realizable but realizable triple contacts may not be semi-free because they are disabled by other constraints. Our analysis will count realizable triple contacts even though our real interest lies in the semi-free ones. However, since we are aiming at an *upper* bound on the number of semi-free triple contacts, such an overcounting is permissible.

Note that the constraint surface in the configuration space that is induced by a single contact of either type is a polygon. Also, the collection of points of the configuration space that represent a pair of contacts is a straight line segment (or the empty set). Three simultaneous contacts induce at most a single point in the configuration space.

By standard arguments (see, e.g., [6, Section 3.1]), it is sufficient to bound the number of vertices on the boundary of FP, in order to get an asymptotic upper bound on the complexity of the boundary of FP.

Some of the vertices on the boundary of FP are obtained by multiple constraints from a single contact, and we refer to such contacts as *degenerate* contacts. A contact between a corner of the box and a vertex of an obstacle will appear as a vertex of an expanded obstacle in the configuration space. A contact between a corner of the box and an edge of an obstacle will appear as an edge of an expanded obstacle, and can therefore contribute to our counting when this edge intersects a face of another expanded obstacle. However, it is easily verified that the overall number of vertices of this type is $O(n^2)$.

We now turn to show the following "warm-up" result.

Lemma 2.1. *The complexity of* FP *is* $\Theta(n^2)$ *if* B *is a rectangle, and* Ω *is a collection of* n *lines in space.*

Proof. *Upper bound.* A non-degenerate contact in this case is an edge contact. If we assume that the obstacle lines are in general position, then any realizable triple contact has two contacts O_1, O_2 that share the same edge of the rectangle or two parallel edges of the rectangle. In either case, if we let the rectangle move while retaining both contacts, it is free to slide along a fixed line parallel to the edge (or edges) involved. If the rectangle retains the contacts O_1, O_2 but is otherwise free to move, then in either direction of the motion along this fixed line, the rectangle can meet at most one more obstacle line, because in order to bypass this obstacle line, it will have to give up the contacts O_1, O_2. Hence, every such pair of contacts induces at most two vertices on the boundary of FP. There are $O(n^2)$ such pairs, and every triple contact must involve such a pair.

Lower bound. The construction consists of two families of lines: (i) $\lceil n/2 \rceil$ vertical lines (i.e., parallel to the z-axis) all lying on the plane $y = 0$ and passing through the planes $x = 1, 2, \ldots, \lceil n/2 \rceil$, and (ii) $\lfloor n/2 \rfloor$ horizontal lines lying in the plane $y = 1$, and passing through the planes $z = 1, 2, \ldots, \lfloor n/2 \rfloor$. We tilt the construction slightly so that no line will be axis parallel. See Fig. 1 for an illustration. Next, consider an axis parallel rectangle, with the appropriate dimensions, such that it can be put inside each of the "holes" created by this grid when looking at it in the y direction, where in each placement the rectangle touches one "horizontal" and one "vertical" line. (The edge of the rectangle touching these lines is taken to be more than one unit long.) This way we get $\Omega(n^2)$ distinct edges on the boundary of free space. □

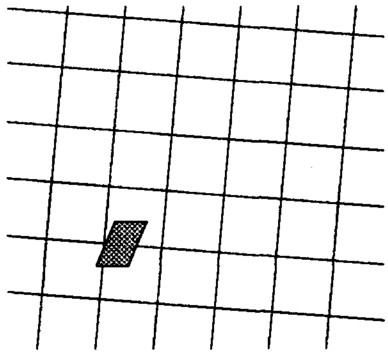

Fig. 1. An $\Omega(n^2)$ lower bound on the complexity of translating a rectangle.

Remarks. (1) Note that the above lower bound is applicable to the main problem that we consider, namely, to the case of a box translating among polyhedral obstacles.

(2) In the above construction, the entire free space consists of one cell. The construction can be easily modified such that the free space will consist of $\Omega(n^2)$ distinct cells.

We conclude this section with a review of a well-known fact, which we will be using throughout the proof of our main result. We start with a definition. We consider a collection A of line segments in the plane, and view each segment as the graph of a partially defined linear function $y = a_i(x)$.

Definition 2.2. The *lower envelope* Ψ of the collection A is the pointwise minimum of these functions: $\Psi(x) = \min a_i(x)$, where the minimum is taken over all functions defined in x. Similarly, the *upper envelope* of the collection A is the pointwise maximum of these functions.

Lemma 2.3. *Let A_1 be a collection of n red segments in the plane and let A_2 be a collection of n blue segments in the plane. The complexity of the lower (respectively upper) envelope of the segments in A_1 (respectively A_2) is $O(n\alpha(n))$. The maximum number of intersections between the blue lower envelope and the red upper envelope is also $O(n\alpha(n))$.*

Proof. By [9], the complexity of the lower (or upper) envelope of n segments is $O(n\alpha(n))$. Thus, the remaining question is how complex is the interaction between these two envelopes. We project the breakpoints of both envelopes onto the x-axis. This will divide the x-axis into $O(n\alpha(n))$ intervals, where the interior of each interval is free from breakpoint projection. Consider one such interval I. Along I, the lower envelope of A_1, and similarly the upper envelope of A_2, is attained by (a portion) of at most one segment. Hence, along I there might be at most one intersection point of the two envelopes. Therefore, the overall number of intersection points of this kind is $O(n\alpha(n))$. □

3. A critical case: translating a triangle

To achieve our main result, we need to analyze a special situation of translating a triangle T amidst the obstacle set Ω. However, we do not pursue the triangle problem in full, but confine ourselves to analyzing only one type of triple contacts for the case of a triangle, which is what we shall need for the case of a box. In Section 5, we will return to the full problem of translating a triangle.

The main result in this section is the following proposition.

Proposition 3.1. *In translating a triangle T, among polyhedral obstacles with a total of n corners there are at most $O(n^2\alpha(n))$ semi-free triple contacts of the form (O_1, O_2, O_3), where $O_i = (c_i, F_i)$ and c_1, c_2, c_3 are the three corners of T.*

To prove this proposition, we follow closely, and adapt to our needs a technique of Leven and Sharir [15], originally devised for bounding the complexity of the free space of a convex body translating and rotating among polygonal obstacles in the plane.

We now set up notations for the proof. Without loss of generality, we assume that the obstacles are a collection of n triangles in 3-space and that the triangle T translates parallel to the xy-plane. Although the original polyhedral obstacles are assumed to be in general position, the present set of triangles derived from these polyhedral obstacles are generally *not* in general position (two triangles can share an edge and several triangles can be coplanar). These violations of the general position will not affect our analysis. However, we stick to the other general position assumptions made in the beginning of Section 2. To further simplify our presentation, we cut each obstacle triangle into two triangles by intersecting it with a plane that is parallel to the xy-plane and passes through the middle corner (middle in z) of the obstacle triangle. Note that the edge formed by the cut is horizontal (we refer to the z direction as vertical). The other two edges can be assumed non-horizontal.

As before, we denote a (potential) [triangle corner, obstacle face] contact by $O_i = (c_i, F_i)$, where c_i is a corner of the moving triangle T and F_i is an obstacle triangle. We regard each triangle as two-sided. By F_i we mean one side of a triangle; we treat each side separately. Consider a placement of the robot triangle T where it makes two simultaneous contacts O_1, O_2. Fig. 2 describes a z-cross-section, where this double contact takes place. Let z_0 be the specific z-value of the placement, and let $F_1(z_0)$ and $F_2(z_0)$ denote the segments, that are the cross-sections of F_1 and F_2 at z_0, respectively. Assume $F_1(z_0)$ and $F_2(z_0)$ are not parallel, and let $\xi(z_0)$ denote the intersection point of the lines containing $F_1(z_0)$ and $F_2(z_0)$. Let $u_i(z_0)$ and $v_i(z_0)$ denote the endpoints of the segment $F_i(z_0)$, where $u_i(z_0)$ is the endpoint closer to $\xi(z_0)$. If $\xi(z_0)$ lies in the interior of either segment, say $F_2(z_0)$, then to avoid ambiguity we denote its endpoints as follows. Let Q be the quadrant defined by the lines containing $F_1(z_0)$ and $F_2(z_0)$ and containing T (in the placement where T simultaneously makes the two contacts O_1, O_2). Then v_2 will denote the endpoint of $F_2(z_0)$ that lies in the closure of Q.

Definition 3.2. Let O_1 and O_2 be two contacts. We say that O_2 *bounds* O_1 at a fixed z-value z_0, if there exists a (not necessarily free) placement (x_0, y_0, z_0) of the robot triangle T which simultaneously satisfies contacts O_1 and O_2, such that the edge $S = \overline{c_1c_2}$ of T always intersects $F_2(z_0)$ as we move T from (x_0, y_0, z_0) without changing $z = z_0$, keeping the contact O_1, in the direction of $\xi(z_0)$, until u_1—the last position at which c_1 still touches $F_1(z_0)$.

The following lemma states the crucial property on which the Leven–Sharir technique relies.

Lemma 3.3. *Let O_1, O_2 be two contacts for which there exists a position (x_0, y_0, z_0) of the robot triangle T which simultaneously satisfies contacts O_1 and O_2. Then, assuming that $F_1(z_0)$ and $F_2(z_0)$*

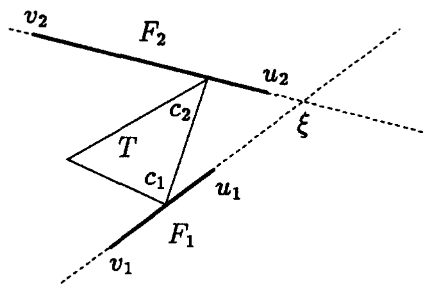

Fig. 2. A double contact with (c_2, F_2) bounding (c_1, F_1).

are not parallel, either O_1 bounds O_2 at z_0 or O_2 bounds O_1 at z_0. Moreover, $[u_1(z_0), u_2(z_0)]$ is parallel to $[c_1, c_2]$ if and only if O_1 and O_2 mutually bound each other.

Proof. Can be taken verbatim from [15, Proposition 2.1, Case (1)]. □

Let O_1 be a contact and consider all other contacts that bound O_1, for some z. For each such contact O_2, we define the function $F_{O_1O_2}(z)$ over the domain $\Pi_{O_1O_2}$ of z values of the placement of T in which O_2 bounds O_1, to be the distance of c_1 from u_1 at the placement $(x, y, z) = (x(z), y(z), z)$ in which T satisfies the two contacts involving O_1, O_2.

To see that $\Pi_{O_1O_2}$ is connected, let $\theta(u, v)$ denote the angle made by the ray from u through v with the positive x-axis (u, v are distinct planar points). Then it is easy to see that the angle

$$\Delta(z) = \theta(c_1(z), c_2(z)) - \theta(u_1(z), u_2(z))$$

is monotonic in z, and O_2 bounds O_1 if and only if $\Delta(z) \geqslant 0$. Connectedness of $\Pi_{O_1O_2}$ follows.

The definition of the endpoint $u_1(z)$ of $F_1(z)$ above, is dependent on the intersection point of the lines containing $F_1(z)$ and $F_2(z)$ relative to the contact point of c_1 and F_1 at z. Hence we partition the collection of bounding functions $F_{O_1O_2}$ for O_1 into two classes A_1 and A_2, where for all functions in A_1, u_1 belongs to the same edge of the obstacle triangle F_1, and similarly for all functions in A_2, u_1 belongs to the other (non-horizontal) edge of F_1. That way, each contact O_1 defines two "complementary" coordinate frames (z, d) which we use to represent placements of T at which it makes an obstacle contact involving O_1. To simplify the analysis, we will fix one (non-horizontal) edge e_1 of each triangle F_1, and describe the functions $F_{O_1O_2}$ in a coordinate frame (z, d), where z is the z-coordinate of the placement of T and d is the distance between the contact point of c_1 and F_1 and the endpoint of $F_1(z)$ that lies on e_1. Let C_{O_1} denote the domain in the coordinate frame (z, d) that represents the contact O_1. Let A_2 be the family of functions $F_{O_1O_2}$ for which the point u_1 lies on e_1. Let A_1 be the family of functions $F_{O_1O_2}$ for which the points u_1 lie on the other edge. Since O_2 bounds O_1, inside C_{O_1} whatever lies above (the graph of) a function in A_1 or below a function in A_2 is not in free space. See Fig. 3 for an illustration.

We now return to the proof of Proposition 3.1.

Proof. Let $P = (x_0, y_0, z_0)$ be a placement of T where there is a triple of corner contacts that appears as a vertex on the boundary of free space. Let $O_i = (c_i, F_i)$, $i = 1, 2, 3$, be the three contacts involved. For each pair $i, j \in \{1, 2, 3\}$, $i \neq j$, either O_i bounds O_j or O_j bounds O_i. It follows that there are three bounding functions $F_{O_iO_j}$, such that on each one of them there is a point corresponding to

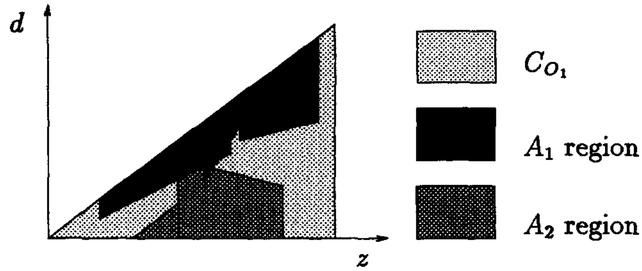

Fig. 3. The functions in A_1 and A_2 and corresponding forbidden regions in the domain C_{O_1}.

the placement P. Suppose, without loss of generality, that it lies on $F_{O_1O_2} \in A_1$. Then, since P is a semi-free placement, $F_{O_1O_2}(z_0)$ is a point on the lower envelope $\phi_{O_1A_1}$ of the functions in A_1. Since at P, the robot T makes the contact O_3 as well, we must have one of the following situations.

(i) O_3 also bounds O_1 and $F_{O_1O_3}$ belongs to A_1. In this case P is represented by a breakpoint of the lower envelope $\phi_{O_1A_1}$.

(ii) O_3 also bounds O_1 but $F_{O_1O_3}$ belongs to A_2. Let $\widetilde{\phi}_{O_1A_2}$ denote the upper envelope of the functions in A_2. In this case P is represented by an intersection point of the lower envelope $\phi_{O_1A_1}$ and the upper envelope $\widetilde{\phi}_{O_1A_2}$.

(iii) No two contacts of O_1, O_2, O_3 bound the third contact at z_0. In this case, we may assume that O_2 bounds O_1, O_3 bounds O_2 and O_1 bounds O_3. We call such a triple contact *circular*.

The number of functions in either A_1 or A_2 is evidently $\mathrm{O}(n)$. All the functions are linear, and therefore the complexity of the envelopes $\phi_{O_1A_1}$, $\widetilde{\phi}_{O_1A_2}$ is $\mathrm{O}(n\alpha(n))$, by Lemma 2.3. The intersection of these two envelopes also has $\mathrm{O}(n\alpha(n))$ points by the same lemma. If we repeat the above analysis for situations (i) and (ii) for every possible contact O_1, then we get that the overall contribution of such placements to the complexity of the boundary of the free space of the motion-planning problem for T is $\mathrm{O}(n^2\alpha(n))$. Let us denote the collection of z-values, at which any envelope (lower or upper) has a breakpoint, by $\mathcal{Z}$. This is a discrete set, by the general position assumption.

To bound the number of placements where situation (iii) occurs, we proceed as follows. We use the set $\mathcal{Z}$ to divide the z-axis into $\mathrm{O}(n^2\alpha(n))$ maximal intervals that do not contain a point of $\mathcal{Z}$ in their interiors. Note that inside each interval, wherever $\phi_{O_iA_1}$ or $\widetilde{\phi}_{O_iA_2}$ are defined, they are attained by a unique function $F_{O_iO_j}$. Consider $z_0 \in \mathcal{Z}$—it corresponds to a breakpoint in one of the envelopes. We show how to "process" each z_0, looking for triple contacts that are circular (the situation (iii)). By the general position assumption, z_0 corresponds to a breakpoint in some envelope E of the form $\phi_{O_iA_1}$ or $\widetilde{\phi}_{O_iA_2}$, for some O_i. In general, the z-axis is divided by E into "z-segments" where each z-segment Π is either "empty" or is determined by a unique function $F_{O_iO_j}$ (for some O_j) whose graph coincides with E in that interval. We say that Π *belongs* to E. A breakpoint of E is called an "end breakpoint" if it is the lower endpoint of an empty z-segment, and a "start breakpoint" (for $F_{O_iO_j}$) otherwise.

Back to our processing of z_0: if z_0 is an end breakpoint of E, we do nothing. Otherwise, z_0 is the start breakpoint for some $F_{O_iO_j}$. Let Π_i be the z-segment belonging to E such that $F_{O_iO_j}$ coincides with E over the range Π_i. This leads us to the obstacle O_j: consider the upper and lower envelopes of O_j. By symmetry, we only discuss the lower envelope $E' = \phi_{O_jA_1}$ since the upper envelope $\widetilde{\phi}_{O_jA_2}$ is similarly treated. Consider the z-segment Π_j belonging to E' that contains z_0. If Π_j is empty, we do nothing; otherwise $\phi_{O_j A_1}$ coincides with $F_{O_j O_k}$ over the range Π_j, for some O_k. This leads us to the third contact O_k. Again there are two z-segments belonging to the lower and upper envelopes of O_k. Focusing on one of the two z-segments Π_k, we can easily check to see if $\Pi_i \cap \Pi_j \cap \Pi_k$ is non-empty. If so, the triple contact (O_i, O_j, O_k) is circular and can be output. This finishes our description for processing z_0. It is also possible that we output the same triple contact (in different order) more than once, but this does no harm for our upper bound. Since there are $O(n^2\alpha(n))$ choices for z_0, we discover at most $O(n^2\alpha(n))$ triples overall.

To show the correctness of this procedure, it is enough to show that every circular triple contact (O_i, O_j, O_k) will be discovered in the course of processing the values in $\mathcal{Z}$. This is because such a triple contact determines three envelopes E_{ij}, E_{jk}, E_{ki} where the functions $F_{O_iO_j}, F_{O_jO_k}, F_{O_kO_i}$ coincide with the respective envelopes. Let the corresponding z-segments determined by these functions be $\Pi_{ij}, \Pi_{jk}, \Pi_{ki}$. Then $\Pi_{ij} \cap \Pi_{jk} \cap \Pi_{ki} = \Pi$ is non-empty. If z_0 is the lower endpoint of Π, then $z_0 \in \mathcal{Z}$ and when we process z_0, we will discover this triple. This concludes the proof of Proposition 3.1. □

4. The case of a box

We now consider the problem of translating a box B among polyhedral obstacles in 3-space. Before attacking this problem, we give a brief overview of the main ideas that we will use both in this section and in the next section (for the case of a convex polygon).

- If O is a contact, let R_O be the space of all translations that realize O. Clearly R_O is a two dimensional set. We can parameterize R_O as a suitable planar set ('patch'). If we assume that the obstacle faces are convex, then R_O is a convex polygon.
- If O' is another contact, the set of all translations in R_O that is non-free by virtue of O' is called the "configuration obstacle" of O' in R_O. The boundary of the configuration obstacle is basically the set of placements that realize the double contact (O, O'). Let this boundary curve be denoted $C_{O,O'}$.
- For each O, we will define a distinguished direction in the patch R_O, designated as "above"; the opposite direction is designated as "below". Relative to this distinguished direction, we define the *upper* and *lower hull* of any curve $C_{O,O'}$ in R_O as follows: a point $p \in R_O$ is in the upper (lower) hull of $C_{O,O'}$ iff it lies above (below) some point of $C_{O,O'}$. The boundary of the upper (lower) hull is called the *upper* (*lower*) *envelope* of $C_{O,O'}$.
- We say that O' *bounds* O if the configuration obstacle of O' in R_O is precisely the upper or lower hull of $C_{O,O'}$. The "bounding property" for a pair O, O' of contacts states that either O bounds O' or O' bounds O. If a triple contact has the property that two of the contacts bound the third, then this triple contact can be "charged" to the third. By the arguments about complexity of envelopes, no contact is charged more than $O(n\alpha(n))$ times, which is a favorable situation. Otherwise, we have a circular triple contact and appeal to the global argument of Section 3.

Back to the case of translating a box. Recall that our plan is to bound the number of triple contacts, that appear as vertices on the boundary of FP. We will count their number by considering three (not necessarily disjoint) sets of triple contacts. The first set is the set of triple contacts, at least one of which is an edge contact. The second set is the set of triple contacts, at least one of which is a face contact. Finally, we will consider the set of triple contacts *all* of which are corner contacts. The analysis of the first two cases is fairly simple (and similar), whereas the analysis of the third case is more involved. However, we have already addressed most of the difficulties of the third case in the previous section.

4.1. The number of triple contacts involving an edge contact

We start with bounding the number of triple contacts involving a fixed edge contact. We will then multiply the resulting bound by $O(n)$ as there are clearly $O(n)$ different edge contacts. We will analyze the case where a side of B, parallel to the z-axis, is in contact with an arbitrary fixed obstacle edge. By symmetry, the analysis for any other edge contact is similar.

Let e be a fixed side of B which is parallel to the z-axis. Let d_z be the length of e. Let E be a fixed edge of an obstacle. For convenience of presentation, we assume that E does not lie on a plane orthogonal to the x-axis; this is another general position assumption. We wish to bound the number of vertices on the boundary of the free space, representing placements where e and E are in contact. This subproblem has two degrees of freedom. We will represent each possible placement P of B, where e and E are in contact, by the pair (r, s) defined as follows: if $\xi \in \mathbb{R}^3$ is the point of contact between e and E when B is in placement P, then r is the x-coordinate of ξ and s is the distance between ξ and the top endpoint of e.

We draw, in the (r, s)-coordinate frame, the constraint curves that represent other contacts of robot features with obstacle features, while the contact (e, E) is maintained. We have transformed a portion of a plane of the original configuration space (the plane containing the constraint surface induced by the contact of e with E), to the (r, s) plane, by a linear transformation, therefore the constraint curves are all straight line segments. Assuming no constraint segment is vertical (which is a part of the general position assumption), each constraint segment has the forbidden region either above it or below it. We partition the constraint curves into two families: A_1 is the collection of constraint segments for which the forbidden region lies above them, and A_2 is the collection of constraint segments for which the forbidden region lies below them. Note that we are only interested in the rectangle of the (r, s) plane corresponding to all placements where the contact between e and E is defined.

The major observation leading to the desired bound is the following lemma.

Lemma 4.1. *If (r_0, s_0) is a point on a constraint curve of the family A_1, then for $s_0 < s_1 \leqslant d_z$, the point (r_0, s_1) does not represent free space.*

Proof. Consider a constraint curve of the family A_1, containing the point (r_0, s_0). This constraint curve expresses the fact that when the edge e of B is in contact with E and it slides in the positive z-direction, it is stopped from above by some other obstacle feature. This obstacle feature must touch the upper face of B at some point. The next free placement of B, in the positive z-direction, must be at least d_z above this stopping point, meaning that the contact (e, E) cannot be retained there. The assertion of the lemma follows. □

The above lemma implies that we are only interested in the lower envelope of the segments of A_1. A break point of this lower envelope, which is a meeting point of two segments, represents a potential triple contact involving the contact (e, E).

By a completely symmetric argument, we are only interested in the upper envelope of the segments in A_2. We are also interested in points where the lower envelope of A_1 meets the upper envelope of A_2.

As for the complexity of these envelopes and their intersection points: each of A_1 and A_2 consists of $\mathrm{O}(n)$ segments. By Lemma 2.3, the complexity of either envelope, as well as the overall number of intersection points of the two envelopes is $\mathrm{O}(n\alpha(n))$.

In summary, the number of triple contacts, one of which is the fixed contact (e, E), is $\mathrm{O}(n\alpha(n))$. If we repeat the analysis for all the $\mathrm{O}(n)$ edge contacts, we obtain the following lemma.

Lemma 4.2. *The number of semi-free triple contacts, in which at least one of the three contacts is an edge contact, is* $\mathrm{O}(n^2\alpha(n))$.

It is interesting to note that the simple analysis above already gives an upper bound on the complexity of the entire free space for the problem of translating a box among n obstacle lines in 3-space. The reason being that in this motion-planning problem, the only possible non-degenerate contact is an edge contact. Thus we have the following corollary.

Corollary 4.3. *The complexity of the entire free space for the motion-planning problem of a box translating among n obstacle lines in 3-space is* $\mathrm{O}(n^2\alpha(n))$.

4.2. The number of triple contacts involving a face contact

We proceed to analyze the second set of triple contacts—the set of those triple contacts at least one of which is a face contact. Recall that a face contact is one that involves a box face touching an obstacle corner. The analysis of the number of triple contacts in this set is similar to the analysis of the previous set. Fix one box face f, which is parallel to the xz-plane, and arbitrarily fix an obstacle corner C, such that (f, C) is a contact pair. We now restrict our attention to this subproblem of motion planning with two degrees of freedom, where f and C are in contact.

We choose the coordinates of the configuration space of this subproblem to be the local coordinates of C in the face f, denoted by (r, s) and defined as follows: when f and C are in contact at the point ξ, r is the (horizontal) distance between ξ and the left edge of f that is parallel to the z direction, and s is the (vertical) distance between ξ and the top edge of f that is parallel to the x-axis. We fix r when B is in some semi-free placement keeping the contact (f, C), and let B slide in the positive z-direction. It will be stopped only when some obstacle feature will touch the top face of B. The next free placement of B, on that plane and having the same r coordinate, must be at least d_z above this stopping point, meaning that the contact (f, C) cannot be retained there. Consequently, we can define two sets of constraint segments A_1, A_2, as in Section 4.1, and proceed exactly as in that case.

In summary we have the next lemma.

Lemma 4.4. *The number of semi-free triple contacts, in which at least one of the three contacts is a face contact, is* $\mathrm{O}(n^2\alpha(n))$.

So far, our analysis has accounted for triple contacts at least one of which is either an edge contact or a face contact. It remains to consider triple contacts consisting only of corner contacts. The bound that we derive for this set is $\mathrm{O}(n^2\alpha(n))$ as well, however, the analysis in this case is more complicated than the analysis in the previous cases.

4.3. *The number of triple corner contacts*

Here we are concerned with triple contacts, each involving three contacts of the form [robot corner, obstacle face]. It is worth noting the following difference between the previous cases and the current case. In each of the two cases that we have already considered, not only have we obtained a near-quadratic bound on the overall contribution of the specific type of triple contacts to the boundary of the free space, but we have in fact derived a near-linear bound on the contribution to the free space involving one fixed contact of the relevant type (e.g., [robot edge, obstacle edge] for the first set of triple contacts that we have considered). It can be easily shown, using a variant of the lower bound construction in Lemma 2.1, that if we fix a pair [robot corner, obstacle face] arbitrarily, the contribution of the triple contacts involving that pair to the boundary of the free space, may be as high as quadratic. Thus, we need a more global argument for the current case.

We fix a triple of distinct corners of the box B and bound the number of triple contacts involving this triple of corners. Evidently, the bound that we will obtain, will serve as an asymptotic upper bound on the number of triple contacts, all of which are corner contacts.

Let T be the triangle which is the convex hull of a fixed triple of corners c_1, c_2, c_3 of B, where B is in some fixed placement in space. Next, consider the free configuration space induced by the problem of translating the triangle T among our original set of polyhedral obstacles. We claim that any vertex v on the boundary of the free space of the original motion-planning problem for B, that involves the corners c_1, c_2, c_3, must show up as a vertex on the boundary of the free space for the problem involving the triangle T. Indeed, in the original problem, the vertex v appears as a vertex on the boundary of the free space, because only the three corners c_1, c_2, c_3 of B are in contact with the obstacles, and no other part of B is in contact with them. Therefore, if we substitute B by T, this triple contact will also appear as a vertex of the free space. The reverse is not true, in general.

This way, we have reduced our subproblem to that of bounding the number of triple contacts all of which are of the form [triangle corner, obstacle face] showing as vertices on the boundary of the free space of a triangle translating among a set of polyhedral obstacles in 3-space, having a total of n features. But, by Proposition 3.1 this bound is $O(n^2\alpha(n))$.

Thus we have proved the main result of the paper.

Theorem 4.5. *For a box B translating among polyhedral obstacles with a total of n features, the complexity of the free space is $O(n^2\alpha(n))$.*

5. Translating a convex polygon

We consider the problem of translating a convex polygon P among polyhedral obstacles in 3-space. We assume that P is horizontal, i.e., that it is parallel to the xy-plane and therefore it translates parallel to the xy-plane (possibly changing its z-coordinate). We fix a horizontal plane Π_0 corresponding to some arbitrary $z = z_0$, and initially restrict the translations Z so that $P[Z]$ is contained in Π_0. We analyze the various double contacts.

I. Two corner contacts, namely O and O' are each a contact between a corner of P and a face of an obstacle. This is just the analysis of Section 3, Lemma 3.3 which states that either O bounds O' or vice versa. In this sense, Section 3 can be viewed as a partial analysis that is now being completed.

II. Two edge contacts and *III. An edge contact and a corner contact.* The proof of the bounding property for each of these cases can be found in [15, Proposition 2.1]. The edge–edge contacts are discussed in Case (2) there, and the edge–corner contacts in Case (3) in their proof.

The preceding discussion of cases I–III depends on some arbitrary $z = z_0$ value. We next show that this dependence on z_0 is not very significant.

Lemma 5.1. *Let (O, O') be a realizable double contact involving edge or corner contacts only. There are $\mathrm{O}(1)$ intervals of z-values such that within any interval, O bounds O' for all z, or O' bounds O for all z.*

Proof. The condition for O bounding O' is a maximum constant degree semialgebraic one, meaning that these condition can be written as a Boolean combination of polynomial inequalities of constant maximum degree. A z-interval corresponds to a connected component of a projection of one such semialgebraic set. But the number of connected components in the projection of such a semialgebraic set is also bounded by a constant depending on the maximum degree of the defining polynomials, as desired. □

This lemma is exploited as follows: as in Section 3, we assume that the patch R_O is parameterized by a pair (z, d), where z is the usual z-coordinate, and the parameter d depends on O. If a contact O' bounds O over $m \geqslant 0$ disjoint intervals of z-values, we introduce m corresponding functions $F_{O,O',i}$ $(i = 1, \ldots, m)$ which are placed in one of two sets A_1, A_2, depending on whether the upper or lower hull of $F_{O,O',i}$ is not in free space. Since $m = \mathrm{O}(1)$, this means that the A_1, A_2 still has $\mathrm{O}(n)$ functions and the corresponding upper ϕ_{O,A_1} and lower $\widetilde{\phi}_{O,A_2}$ envelopes have complexity $\mathrm{O}(n\alpha(n))$.

We may conclude the following lemma.

Lemma 5.2. *The number of realizable triple contacts that involve only edge or corner contacts of P is $\mathrm{O}(n^2\alpha(n))$.*

Proof. Let (O_1, O_2, O_3) be a realizable triple contact. If one contact (say O_1) is bounded by the other two, then this appears as a vertex in the union of the upper and lower envelope of curves in R_{O_1}. Using the "envelope" argument (situations (i) and (ii) in Section 3), there are $\mathrm{O}(n\alpha(n))$ such vertices in R_{O_1}. In the contrary case, we may assume that O_1 bounds O_2 which bounds O_3 which in turn bounds O_1. In this case, the global argument in Section 3 will work. □

In the current setting we still need to consider one additional type of contacts.

IV. Face-corner contacts. Let $O = (f, C)$ be a contact between a face of P and a corner of the obstacles. Since our robot is a polygon P, there are just two choices for f, corresponding to the top and bottom sides of P (as in Section 3 we view P as two-sided). Without loss of generality, assume that C is the origin of the horizontal plane $\{z = 0\}$ and P lies in the same plane. By non-degeneracy, there are no other corners in the plane $\{z = 0\}$.

Let R_O denote the polygonal patch corresponding to placements that satisfy O. For any obstacle set S that intersects the plane $\{z = 0\}$, there is the corresponding *configuration obstacle* S' corresponding to all placements in R_O which is non-free by virtue of S.

We have now reduced the problem of bounding the number of contacts involving the face–corner contact (f, C), to the problem of bounding the complexity of the free space of a convex polygon translating among polygonal obstacles with a total of at most n features in the plane. It follows from [11] that for a fixed size translating polygon, this complexity is $O(n)$. Repeating this argument for each corner of any obstacle in our setting, and for each of the two faces of P, we obtain that the overall number of vertices on the boundary of the free space involving a face–corner contact, is $O(n^2)$.

The lower bound $\Omega(n^2\alpha(n))$ mentioned in Section 1 holds for a convex polygon. Thus our bound in this case is tight.

Theorem 5.3. *For a convex polygon P, with a fixed number of sides, translating among polyhedral obstacles with a total of n features, the complexity of the free space is $\Theta(n^2\alpha(n))$.*

6. Conclusion and open problems

We have shown that the complexity of the free space for a box translating in 3-space among polyhedral obstacles with a total of n features is $O(n^2\alpha(n))$. The same bound holds if we substitute the box with a convex polygon of fixed size. The bound for the box is within an $\alpha(n)$ factor off the lower bound for this case. The bound for a convex polygon is tight, i.e., there is a construction where this bound is obtained.

Our new bounds constitute an improvement of almost an order of magnitude over the best previously known (and naive) bound for this problem, $O(n^3)$. As mentioned in Section 1, considerable advancement on the related general problem has been recently obtained by Aronov et al. [2,3]. Our techniques, for the special cases that we handle, are fairly simple, and our bounds are sharper (and in fact tight for the case of a convex polygon).

The paper raises several problems for further study.

- An intriguing open problem in this area is to obtain a non-trivial bound on the complexity of the free space for a ball moving among obstacles in 3-space, say even line obstacles. For this problem, there is still a gap of an order of magnitude between the lower (quadratic) and upper (cubic) bounds. An equivalent formulation of this problem concerns the complexity of the boundary of the union of infinite congruent cylinders in 3-space. Here also, it is a prevalent conjecture, that the actual bound is (roughly) quadratic.
- Another open problem is to obtain a sharp upper bound (ideally, $O(n^2\alpha(n))$) when B is an arbitrary convex polyhedron. (The result [3] mentioned in Section 1, still has a logarithmic factor in the upper bound.)
- Is the upper bound for translating a box really only $\Theta(n^2)$?

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